

Deformation of Thin Cylindrical Shells
Subjected to Internal Loading

by
J. A. Rojlasrak

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1. Introduction

In this paper the problem of deformation of a circular cylindrical shell freely supported at its edges and subjected to internal hydrostatic pressure is discussed. Trigonometric series for expressing displacements have been used¹ and it was shown that the coefficients of these series rapidly diminish so that only few terms are necessary to represent the deformation with sufficient accuracy. General equations were applied to numerical examples.

2. General Equations

Considering the condition of equilibrium of an element of the cylindrical shell cut out by two diametral sections and two cross sections perpendicular to the axis of the cylindrical shell and using for stress-resultants and stress-couples the notation indicated in Fig. 1² where the vectors representing the stress-couples H_1 and G_1 have the same directions as T_1 and S_1 and the vectors representing the stress-couples H_2 and G_2 have the same directions as T_2 and S_2 , the following equations of equilibrium are obtained:

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1. This method has been used previously by S. Timoshenko, see: Theory of Elasticity, Vol. 2, pp. 384-387, St. Petersburg (1916). See also: H. Reissner, Zeit. f. ang. Math. u. Mech., Vol. 13, p. 133 (1933).
 2. See A.E.H. Love, The Math. Theory of Elasticity (1927), pp. 534-536.

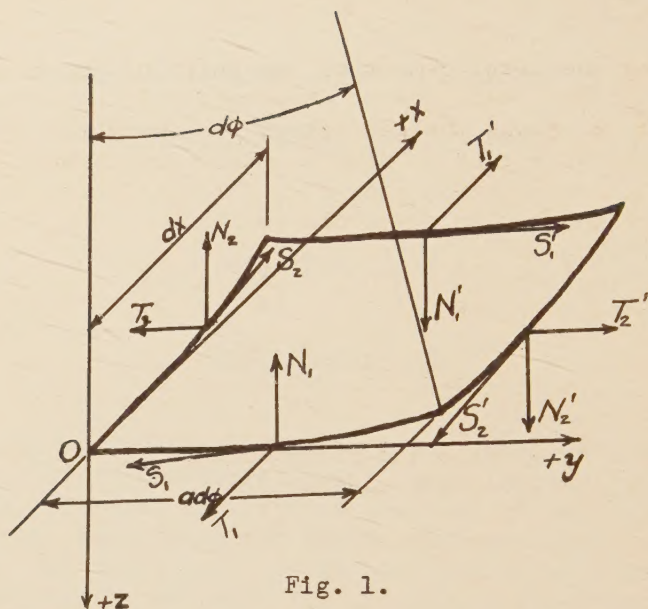


Fig. 1.

$$\begin{aligned}
 a \frac{\partial T_1}{\partial x} - \frac{\partial S_2}{\partial \phi} &= 0 & a \frac{\partial H_1}{\partial x} - \frac{\partial G_2}{\partial \phi} + N_2 a &= 0 \\
 a \frac{\partial S_1}{\partial x} + \frac{\partial T_2}{\partial \phi} + N_2 &= 0 & a \frac{\partial G_1}{\partial x} + \frac{\partial H_2}{\partial \phi} - N_1 a &= 0 \quad (1) \\
 a \frac{\partial N_1}{\partial x} + \frac{\partial N_2}{\partial \phi} - T_2 + qa &= 0 \quad (S_1 + S_2)a = 0
 \end{aligned}$$

where q is the load per unit area of the middle surface acting in the direction of the positive z -axis.

Eliminating N_1 and N_2 from eqs. (1) and using the relations:

$$S_1 = -S_2 = S \text{ and } H_1 = -H_2 = H$$

we obtain the following eqs. of equilibrium:

$$\begin{aligned}
 a \frac{\partial T_1}{\partial x} + \frac{\partial S}{\partial \phi} &= 0 \\
 a \frac{\partial S}{\partial x} + \frac{\partial T_2}{\partial \phi} + \frac{\partial G_2}{\partial \phi} - \frac{\partial H}{\partial x} &= 0 \quad (2)
 \end{aligned}$$

$$\frac{\partial^2 G_1}{\partial x^2} - \frac{2}{a} \frac{\partial^2 H}{\partial x \partial \phi} + \frac{1}{a^2} \frac{\partial^2 G_2}{\partial \phi^2} - \frac{T_2}{a} + q = 0$$

Resolving the displacement of any point of the middle surface of the shell during deformation into three components: u along the generator, v along the tangent to the circular section and w along the normal to the surface drawn outwards (see Fig. 2), the strain components and changes in curvature are:¹

$$\epsilon_1 = \frac{\partial u}{\partial x}; \quad \epsilon_2 = \frac{1}{a} \frac{\partial v}{\partial \phi} + \frac{w}{a}; \quad \omega = \frac{\partial v}{\partial x} + \frac{1}{a} \frac{\partial u}{\partial \phi} \quad (3)$$

$$k_1 = \frac{\partial^2 w}{\partial x^2}; \quad k_2 = \frac{1}{a^2} \frac{\partial^2 w}{\partial \phi^2} - \frac{1}{a^2} \frac{\partial v}{\partial \phi}; \quad \tau = \frac{1}{a} \frac{\partial^2 w}{\partial x \partial \phi} - \frac{1}{a} \frac{\partial v}{\partial x}$$

The stress-resultants and stress-couples are represented now by the following equations:²

$$T_1 = \frac{2Eh}{1-\sigma^2} (\epsilon_1 + \sigma \epsilon_2); \quad T_2 = \frac{2Eh}{1-\sigma^2} (\epsilon_2 + \sigma \epsilon_1);$$

$$S = \frac{Eh}{1+\sigma} \omega \quad (4)$$

$$G_1 = -D(k_1 + \sigma k_2); \quad G_2 = -D(k_2 + \sigma k_1); \quad H = D(1-\sigma)\tau$$

where $2h$ is the thickness of the shell and

$$D = \frac{2}{3} \frac{Eh^3}{1-\sigma^2} = \text{Flexural Rigidity.}$$

Substituting this into (2) and neglecting $h^2/3a^2$ in comparison with $1/2$ and 1 in the second equation we obtain:

$$\frac{\partial^2 u}{\partial x^2} + \frac{1-\sigma}{2a^2} \frac{\partial^2 u}{\partial \phi^2} + \frac{1+\sigma}{2a} \frac{\partial^2 v}{\partial x \partial \phi} + \frac{\sigma}{a} \frac{\partial w}{\partial x} = 0$$

$$\frac{1+\sigma}{2} \frac{\partial^2 u}{\partial x \partial \phi} + \frac{a(1-\sigma)}{2} \frac{\partial^2 v}{\partial x^2} + \frac{1}{a} \frac{\partial^2 v}{\partial \phi^2} + \frac{1}{a} \frac{\partial w}{\partial \phi} - \frac{h^2}{3a} \times$$

$$\times \left(\frac{\partial^3 w}{\partial x^2 \partial \phi} + \frac{1}{a^2} \frac{\partial^3 w}{\partial \phi^3} \right) = 0 \quad (5)$$

$$\sigma a \frac{\partial u}{\partial x} + \frac{\partial v}{\partial \phi} - \frac{h^2}{3} \left\{ (2-\sigma) \frac{\partial^3 v}{\partial x^2 \partial \phi} + \frac{1}{a^2} \frac{\partial^3 v}{\partial \phi^3} \right\} + w +$$

$$+ \frac{h^2}{3} \left\{ a^2 \frac{\partial^4 w}{\partial x^4} + \frac{2 \partial^4 w}{\partial x^2 \partial \phi^2} + \frac{\partial^4 w}{a^2 \partial \phi^4} \right\} = - \frac{qa^2 h^2}{3D}$$

1. See A.E.H. Love, loc. cit. ante pp. 521, 524, and 568.

2. Ibid., p. 530.

These three equations will now be applied to particular cases.

3. Circular Cylindrical Shell Under Internal Hydrostatic Pressure

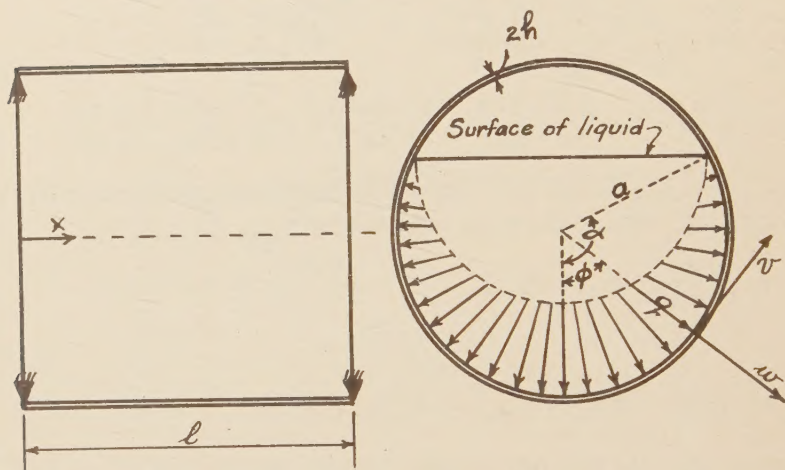


Fig. 2.

In the case of a circular cylindrical shell freely supported at the edges as shown in Fig. 2 and submitted to the pressure of liquid within the cylindrical shell, we have the intensity of the load q represented by the following equations:

$$\begin{aligned} q &= \gamma a (\cos \varphi - \cos \alpha), \quad \text{when } 0 \leq \varphi \leq \alpha \\ q &= 0 \quad \text{when } \pi \geq \varphi \geq \alpha \end{aligned} \quad (6)$$

where γ is the weight of the liquid per unit volume and angles α and φ are shown in Fig. 2. The conditions of symmetry and the conditions at the supports will be satisfied if we take the components of displacement in the form of the following series:

$$\begin{aligned} u &= \sum \sum A_{mn} h \cos n\varphi \cos \frac{m\pi x}{l} \\ v &= \sum \sum B_{mn} h \sin n\varphi \sin \frac{m\pi x}{l} \\ w &= \sum \sum C_{mn} h \cos n\varphi \sin \frac{m\pi x}{l} \end{aligned} \quad (7)$$

The load q also can be represented by the series:

$$q = \sum_m \sum_n D_{mn} \cos n\varphi \sin \frac{m\pi x}{l} \quad (8)$$

The coefficients D_{mn} which can be readily calculated in the usual way from (6) are:

$$D_{mn} = \frac{8\gamma a}{m\pi^2(n^2 - 1)} \left[\cos \alpha \sin n\alpha - n \cos n\alpha \sin \alpha \right] \quad (9)$$

where $m = 1 \cdot 3 \cdot 5 \dots$, and $n = 2 \cdot 3 \cdot 4 \dots$, while,

$$D_{m0} = \frac{4\gamma a}{m\pi^2} (\sin \alpha - \cos \alpha) \quad (10)$$

and

$$D_{m1} = \frac{2\gamma a}{m\pi^2} (2\alpha - \sin 2\alpha) \quad (11)$$

In the case of a cylindrical shell completely filled with liquid we denote by $z\gamma$ the pressure at the axis of the cylindrical shell (see Fig. 3), then:

$$q = \gamma(z + a \cos \varphi) \quad (12)$$

and we obtain instead of (9), (10), (11):

$$\begin{aligned} D_{mn} &= 0 \\ D_{m0} &= \frac{4\gamma z}{m\pi} \\ D_{m1} &= \frac{4\gamma a}{m\pi} \end{aligned} \quad (13)$$

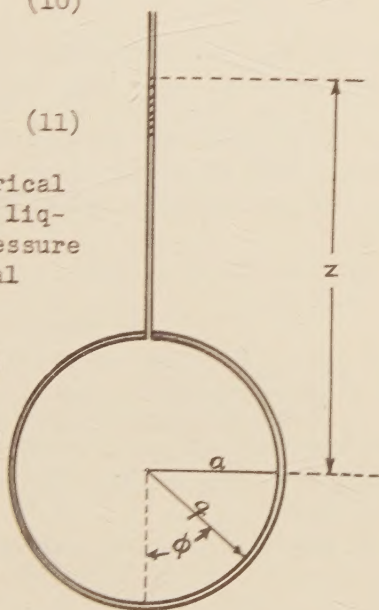


Fig. 3.

Substituting (7) and (8) into (5) and introducing the notation:

$$\lambda = \frac{l}{a}; \quad \eta = \frac{h}{l}$$

we obtain:

$$A_{mn} \left[2m^2\pi^2 + (1 - \sigma) \lambda^2 n^2 \right] - B_{mn}(1 + \sigma) \lambda_{mn\pi} - C_{mn} 2\sigma \lambda_{mn\pi} = 0$$

$$A_{mn} 3(1 + \sigma) \lambda_{mn\pi} - B_{mn} \left[3(1 - \sigma)m^2\pi^2 + 6\lambda^2 n^2 \right] - C_{mn} 2\lambda^2 n \left[3 + \eta^2 (m^2\pi^2 + \lambda^2 n^2) \right] = 0 \quad (14)$$

$$A_{mn} 3\sigma \lambda_{mn\pi} - B_{mn} \lambda^2 n \left[3 + \eta^2 \left\{ (2 - \sigma)m^2\pi^2 + \lambda^2 n^2 \right\} \right] - C_{mn} \left[3\lambda^2 + \eta^2 (m^2\pi^2 + \lambda^2 n^2)^2 \right] = - \frac{D_{mn} l^2 h}{D}$$

Since D_{mn} is known from eqs. (9), (10), (11), or (13) coefficients A_{mn} , B_{mn} , C_{mn} can be calculated in each particular case from (14) for any value of m and n and therefore the stress-resultants, stress-couples and components of displacement can be found for any point of the cylindrical shell by using eqs. (3), (4), and (7).

4. Numerical Examples

The above theoretical solution will now be applied to the case of a cylindrical shell full of liquid and pressure p at the axis and with the following numerical data:

$$a = 50. \text{ cm. ; } l = 25. \text{ cm.}$$

$$h = 3.5 \text{ cm. ; } \sigma = .3$$

Then:

$$\alpha = \pi$$

$$\lambda = \frac{l}{a} = .5 ; \lambda^2 = .25$$

$$\eta = \frac{h}{l} = .14 ; \eta^2 = .0196$$

From (13) we obtain

$$D_{m0} = D_{m1} = \frac{4\gamma a}{m\pi}, \text{ where } m = 1 \cdot 3 \cdot 5 \dots$$

and $D_{mn} = 0$, for $n > 1$.

Beginning with the terms $n = 0$; $m = 1 \cdot 3 \cdot 5 \dots$ of the series (7) and using the notation:

$$\psi = \frac{4\gamma a}{\pi^2} \cdot \frac{\rho^2 h}{D}$$

the solution of eqs. (14) is found to be:

$$\begin{aligned} A_{m0} &= \frac{\psi \lambda \sigma}{3m^2 \left[\lambda^2 (1 - \sigma^2) + \frac{\eta^2}{3} m^4 \pi^4 \right]} \\ B_{m0} &= 0 \\ C_{m0} &= \frac{\psi \pi}{3m \left[\lambda^2 (1 - \sigma^2) + \frac{\eta^2}{3} m^4 \pi^4 \right]} \end{aligned} \quad (15)$$

From eqs. (15) A_{m0} , B_{m0} , and C_{m0} may be found for any value of m . These values, for $m = 1, 3$, and 5 , are compiled in Table 1.

In order to find the coefficients A_{m1} , B_{m1} , C_{m1} we put $n = 1$ in eqs. (14) and find, after letting $A_{m1}' = A_{m1} \cdot m$, from the first of eqs. (14) that:

$$C_{m1} = A_{m1}' \left[20.94 + .1857 \frac{1}{m^2} \right] - B_{m1} 2.167 \quad (16)$$

Eliminating C_{m1} from the last two of eqs. (14) and using the notation:

$$\begin{aligned} P_{m1} &= m^2 (25.36 + 2.026 m^2) + .2790 \\ q_{m1} &= m^2 (1.755 - 20.52 m^2) \\ r_{m1} &= 14.29 m^2 + .1393 + (.4105 m^2 + .003639) \times \\ &\quad \times (.25 + 9.870 m^2)^2 \\ S_{m1} &= m^2 \left[.8738 - .08221 m^2 + .04247 (.25 + \right. \\ &\quad \left. + 9.870 m^2)^2 \right] \end{aligned} \quad (17)$$

TABLE 1

n = 0

m	$A_{m0} \psi^{-1} \times 10^3$	$B_{m0} \psi^{-1} \times 10^3$	$C_{m0} \psi^{-1} \times 10^3$	$A_{m0}' \psi^{-1} \times 10^3$
1	57.88	0	1212.	57.88
3	.1073	0	6.742	.3219
5	.005025	0	.5263	.0251

TABLE 2

n = 1

m	p	q	r	S	πm	$A_{m1}' \psi^{-1} \times 10^3$	$B_{m1} \psi^{-1} \times 10^3$	$C_{m1} \psi^{-1} \times 10^3$
1	27.66	-18.76	56.84	5.140	3.142	48.76	-71.90	1186.
3	392.6	-1646.	29470.	3034.	9.425	.3121	-.07444	6.705
5	1900.	-12780.	626600.	64740.	15.71	.02469	-.003671	.5252

we find the following equations for calculating A_{m1}' and B_{m1}' :

$$\begin{aligned} A_{m1}' p_{m1} - B_{m1}' q_{m1} &= 0 \\ A_{m1}' r_{m1} - B_{m1}' S_{m1} &= \psi \pi m \end{aligned} \quad (18)$$

For any value of m we may first calculate p_{m1} , q_{m1} , r_{m1} , S_{m1} , from eqs. (17) and then the coefficients A_{m1}' , B_{m1}' are found from (18) after which C_{m1} is found from (16). Values of $A_{m1}' \psi^{-1} \times 10^3$, $B_{m1}' \psi^{-1} \times 10^3$ and $C_{m1} \psi^{-1} \times 10^3$, for $m = 1, 3$, and 5 are compiled in Table 2.

Having the coefficients A_{m0} , A_{m1} C_{m1} the stress-resultants and stress-couples will be calculated as follows: From eqs. (3), (4), and (7) it is seen that

$$T_2 = \frac{3D\psi}{\ell h} \sum \sum T_{mn} \cos n\phi \sin \frac{m\pi x}{\ell} \quad (19)$$

and,

$$G_2 = \frac{Dh\psi}{\ell^2} \sum \sum G_{mn} \cos n\phi \sin \frac{m\pi x}{\ell}$$

where:

$$\begin{aligned} T_{mn} &= \psi^{-1} \left[-A_{mn} \sigma m \pi + B_{mn} \lambda n + C_{mn} \lambda \right] \\ G_{mn} &= \psi^{-1} \left[B_{mn} \lambda^2 n + C_{mn} (\sigma m^2 \pi^2 + \lambda^2 n^2) \right] . \end{aligned}$$

Values of $T_{mn} \times 10^3$ and $G_{mn} \times 10^3$, for $n = 0, 1$, and $m = 1, 3, 5$ are compiled in Table 3.

TABLE 3

	$m = 1$		$m = 3$		$m = 5$	
n	$T_{1n} \times 10^3$	$G_{1n} \times 10^3$	$T_{3n} \times 10^3$	$G_{3n} \times 10^3$	$T_{5n} \times 10^3$	$G_{5n} \times 10^3$
0	551.5	3589.	3.068	179.7	.2394	38.95
1	511.2	3791.	3.021	180.3	.2375	39.01

Maximum values of w , T_2 , G_2 , T_1 and G_1 occur at $x = \ell/2$, $\phi = 0$ and will now be computed. Substituting in the third of eqs. (7)

$$x = \frac{\ell}{2}, \phi = 0, m = 1, 3 \text{ and } 5, \text{ and } n = 1, 1$$

we obtain:

$$w_{\max} = h(C_{10} + C_{11} - C_{30} - C_{31} + C_{50} + C_{51})$$

Similarly from (19):

$$T_{2\max} = \frac{3D\Psi}{\ell h} (T_{10} + T_{11} - T_{30} - T_{31} + T_{50} + T_{51})$$

and

$$G_{2\max} = \frac{Dh\Psi}{\ell^2} (G_{10} + G_{11} - G_{30} - G_{31} + G_{50} + G_{51}).$$

In the same manner we obtain:

$$G_1 = \frac{D\Psi h}{\ell^2} \sum \sum G'_{mn} \cos n\phi \sin \frac{m\pi x}{\ell}$$

$$T_1 = \frac{3D\Psi}{h\ell} \sum \sum T'_{mn} \cos n\phi \sin \frac{m\pi x}{\ell}$$

where:

$$G'_{mn} = \Psi^{-1} \left[B_{mn} \sigma \lambda^2 n + C_{mn} (m^2 \pi^2 + \sigma \lambda^2 n^2) \right]$$

$$T'_{mn} = \Psi^{-1} \left[-A_{mn} m\pi + B_{mn} \sigma \lambda n + C_{mn} \lambda \sigma \right]$$

Values of G'_{mn} and T'_{mn} , for $n = 0, 1$ and $m = 1, 3, 5$, are given in Table 4.

TABLE 4

n	m = 1		m = 3		m = 5	
	$T'_{1n} \times 10^3$	$G'_{1n} \times 10^3$	$T'_{3n} \times 10^3$	$G'_{3n} \times 10^3$	$T'_{5n} \times 10^3$	$G'_{5n} \times 10^3$
0	0	11960.	0	598.9	0	129.9
1	13.93	11790	-1.946	596.1	-.309	129.7

With the aid of the above equations and Tables 1 to 4 inclusive, we find:

$$w_{\max} = 11,790. \frac{\gamma}{E} \text{ cm. ; } T_{2\max} = 1.607 \gamma \text{ kg./cm.}$$

$$G_{2\max} = 1.762 \gamma \text{ kg. ; } T_{1\max} = .02367 \gamma \text{ kg/cm.}$$

$$G_{1\max} = 5.663 \gamma \text{ kg.}$$

$$\text{Max. circum. stress} = \frac{T_{2\text{max}}}{2h} + \frac{3}{2} \frac{G_{2\text{max}}}{h^2} = .4453 \gamma \text{ kg/cm.}^2$$

$$\text{Max. long. stress} = \frac{T_{1\text{max}}}{2h} + \frac{3}{2} \frac{G_{1\text{max}}}{h^2} = .6968 \gamma \text{ kg./cm.}^2$$

It is seen that for the cylindrical shell chosen the resulting maximum stresses for the case of the cylindrical shell full of liquid are negligible from the practical point of view. Due to the comparatively large value of $G_{1\text{max}}$ the maximum longitudinal stress is greater than the maximum circumferential stress.

If the cylindrical shell full of liquid is subjected to a pressure $z\gamma$ at its axis (see Fig. 3), then find eqs. (13)

$$D_{m0} = \frac{4\gamma z}{m\pi}; \quad D_{m1} = \frac{4\gamma a}{m\pi}$$

while $D_{mn} = 0$, for $n > 1$.

Since the value of D_{m1} for this case is the same as the value of D_{m1} for the case of the cylindrical shell full of liquid and pressure $a\gamma$ at its axis we may use Table 2 to obtain the coefficients: $A_{m1}' \times \psi^{-1} \times 10^3$, $B_{m1} \psi^{-1} \times 10^3$, $C_{m1} \times \psi^{-1} \times 10^3$. Also since

$$D_{m0} = \frac{4\gamma z}{m\pi} = \frac{z}{a} \cdot \frac{4\gamma a}{m\pi} \quad \text{we may use Table 1 to obtain}$$

the coefficients:

$$A_{m0} \left(\frac{z}{a} \psi\right)^{-1} \times 10^3, \quad B_{m0} \left(\frac{z}{a} \psi\right)^{-1} \times 10^3 \quad \text{and}$$

$$C_{m0} \left(\frac{z}{a} \psi\right)^{-1} \times 10^3.$$

It is seen that the coefficients $A_{m0} \psi^{-1} \times 10^3$ and $C_{m0} \psi^{-1} \times 10^3$ for this case will be z/a times those given in Table 1. Also the coefficients $T_{m0} \times 10^3$ and $G_{m0} \times 10^3$ will be z/a times those given in Table 3 while $T_{m1} \times 10^3$ and $G_{m1} \times 10^3$ will remain the same as those given in Table 3.

The maximum values of T_2 , G_2 , T_1 , and G_1 occur at $x = l/2$ and $\phi = 0$. For the case where $z/a = 100$, they are:

$$T_{\theta\max} = 84.17 \gamma \text{ kg./cm.} ; G_{\theta\max} = 86.49 \gamma \text{ kg.}$$

$$T_{z\max} = .02367 \gamma \text{ kg./cm.} ; G_{z\max} = 296. \gamma \text{ kg.}$$

From which the maximum stresses are found to be:

$$\text{Max. Circum. stress} = 22.61 \gamma \text{ kg./cm.}^2$$

$$\text{Max. long. stress} = 35.27 \gamma \text{ kg./cm.}^2$$

It is seen that the maximum longitudinal stress is also the maximum stress in the cylindrical shell which we are considering.

In examining Tables 1 to 4 inclusive it is observed that the coefficients $A_{mn} \dots G_{mn}$ diminish rapidly so that the series representing the stress-resultants, stress-couples, and components of displacement evaluated on the basis of $m = 0, 1$, and $n = 1, 2, 3$ give fairly accurate results. However, in the case of a long thin cylindrical shell having say, $\lambda = 8$ and $\mu = 1/800$ it is found that the coefficients $A_{mn} \dots G_{mn}$ diminish rather slowly so that it is necessary to compute a somewhat large number of coefficients in order that the series representing the stress-resultants, stress-couples and components of displacement when evaluated will yield results of sufficient accuracy.

